

Weak–strong clustering transition in renewing compressible flows

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We investigate the statistical properties of Lagrangian tracers transported by a time-correlated compressible renewing flow. We show that the preferential sampling of the phase space performed by tracers yields significant differences between the Lagrangian statistics and its Eulerian counterpart. In particular, the effective compressibility experienced by tracers has a non-trivial dependence on the time correlation of the flow. We examine the consequence of this phenomenon on the clustering of tracers, focusing on the transition from the weak- to the strong-clustering regime. We find that the critical compressibility at which the transition occurs is minimum when the time correlation of the flow is of the order of the typical eddy turnover time. Further, we demonstrate that the clustering properties in time-correlated compressible flows are non-universal and are strongly influenced by the spatio-temporal structure of the velocity field.

I. INTRODUCTION

The dynamics of tracers in turbulent flows has important applications in a variety of physical phenomena ranging from the dispersion of atmospheric pollutants [13] to the transport of plankton in the oceans [1]. Moreover, the motion of tracers is intimately related to the mixing properties of turbulent flows and therefore determines the statistics of passive fields such as temperature in a weakly heated fluid or the concentration of a dye in a liquid [17]. The last fifteen years have seen a renewed interest in the Lagrangian study of turbulence thanks to the development of new experimental and numerical particle-tracking techniques [31, 38].

Several of the qualitative properties of tracer dynamics in turbulent flows have been understood by means of the Kraichnan [23] model, in which the velocity is a homogeneous and isotropic Gaussian field with zero correlation time and power-law spatial correlations. Under these assumptions, the separations between tracers form a multi-dimensional diffusion process with space-dependent diffusivity, the properties of which have been studied analytically [9, 17]. In particular, in the smooth and incompressible regime of the Kraichnan model, tracers behave chaotically and spread out evenly in the fluid. If the velocity field is weakly compressible, the Lagrangian dynamics remains chaotic, but tracers cluster over a fractal set with Lyapunov dimension $1 < D_L < d$, where d is the spatial dimension of the fluid [11, 26, 27]. The Lyapunov dimension decreases as the degree of compressibility increases; furthermore, the density of tracers exhibits a multifractal behaviour in space, which indicates the presence of strong fluctuations in the distribution of tracers within the fluid [4]. Finally, if the degree of compressibility of the Kraichnan model exceeds a critical value, all the Lyapunov exponents of the flow become negative and tracers collapse onto a pointlike fractal with $D_L = 0$. This latter regime only exists if $d \leq 4$ [11] and is known as the regime of strong compressibility.

In nature, compressible flows are found not only for large values of the Mach number. An example of a low-Mach-number compressible flow is given by the velocity field on the free surface of a three-dimensional incompressible flow [12, 36]. Furthermore, at small Stokes numbers, the dynamics of inertial particles in an incompressible flow can be assimilated to that of tracers in an effective compressible velocity field [8, 29]. Compressible flows like those mentioned above have a nonzero correlation time. In this respect, the Kraichnan model is not realistic and quantitative discrepancies may be expected between the theoretical predictions and the experimental and numerical observations. In a numerical simulation of a turbulent surface flow, Boffetta *et al.* [6] have found that D_L does decrease as a function of the degree of compressibility, in accordance with the prediction of the Kraichnan model. However, the transition from the weak- to the strong-clustering regime occurs at a larger degree of compressibility compared to the Kraichnan model. This phenomenon is counterintuitive, because the level of clustering would be expected to increase when the correlation of the flow is nonzero. Thus, the study by [6] raises the questions of the interplay between compressibility and temporal correlation in turbulent flows and of the degree of universality of the weak–strong clustering transition [for further studies on clustering in turbulent surface flows, see 7, 14, 24, 25, 28, 32, 35, 39]. The investigation of the universality of this phenomenon is particularly interesting in the light of previous findings of non-universal transport properties in random compressible flows for passive scalar fields [15].

For inertial particles, the effect of the temporal correlation of the flow on the Lagrangian dynamics has been studied analytically in the following one-dimensional cases: for a velocity gradient described by the telegraph noise [19] or by the Ornstein–Uhlenbeck process [40] and for a velocity field given by a Gaussian potential with exponential correlations [21]. In the case of tracers in compressible flows, Chaves *et al.* [10] have studied two-particle dispersion in Gaussian self-similar random fields. Falkovich & Martins Afonso [18] have calculated the Lyapunov exponent

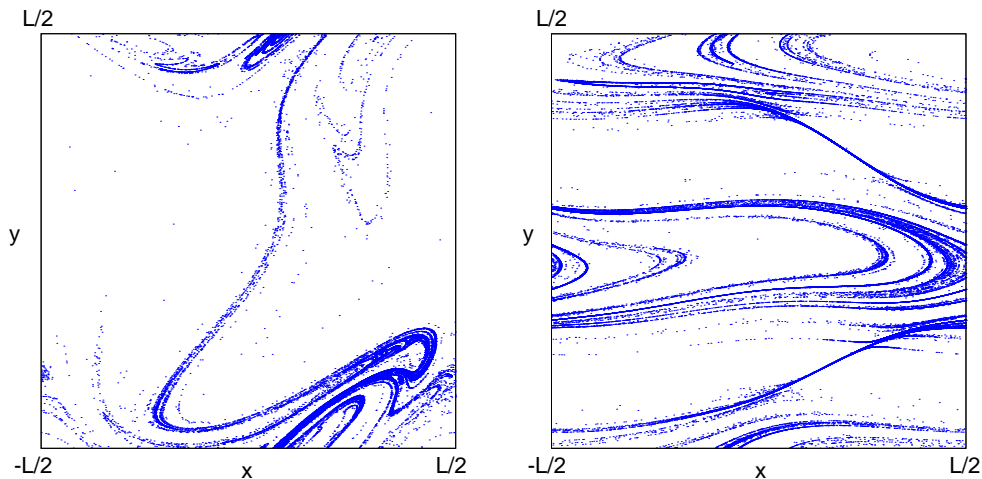


FIG. 1: (Colour online) Spatial distribution of tracers in the compressible renewing flow for $C = 1/4$ and $Ku = 0.1$ (left), $Ku = 10$ (right). Each plot shows the positions of 10^5 tracers. For $Ku = 10$, the distribution of tracers mirrors the structure of the flow, which consists of periodic channels and of regions where transport is inhibited by the stagnation lines.

and the statistics of the stretching rates for a one-dimensional strain described by the telegraph noise. Gustavsson & Mehlig [22] have obtained the Lyapunov exponents of a two-dimensional random flow in the limits of short and long correlation times; the solenoidal and potential components of the velocity were assumed to be Gaussian random functions with exponential spatio-temporal correlations.

In this paper, we undertake a thorough study of the effects of temporal correlations on tracer dynamics in a compressible random flow. We consider a compressible version of the two-dimensional renewing flow, which consists of a random sequence of sinusoidal velocity profiles with variable origin and orientation. Each profile remains frozen for a fixed time; by changing the duration of the frozen phase, we can vary the correlation time of the flow and examine the effect on clustering. The renewing flow, in its original incompressible version, has been successfully applied to the study of the kinematic dynamo [20, 42], of chaotic mixing [2, 3, 15, 34, 37, 41], of inertial-particles dynamics [16, 33] and of polymer stretching [30]. The properties of this model flow allow us to fully characterise the Lagrangian statistics of tracers as a function of the degree of compressibility and for a wide range of correlation times. We show that, in a time-correlated compressible flow, even single-time Lagrangian averages can differ considerably from their Eulerian counterparts. Furthermore, we demonstrate that the properties of clustering do not depend only on universal parameters such as the degree of compressibility and the Kubo number, but also on the specific spatial and temporal properties of the velocity field. In particular, we show that a crucial role is played by the spatial distribution of the stagnation points.

The rest of the paper is organized as follows. In § II, we introduce the compressible renewing flow and describe its Eulerian properties. In § III, we compare the Lagrangian and Eulerian statistics of tracer dynamics as a function of the degree of compressibility and of the correlation time of the flow. Section IV describes the fractal clustering of tracers in the weakly compressible regime and the weak–strong clustering transition. Section V concludes the paper by discussing the non-universal character of the weak–strong clustering transition.

II. COMPRESSIBLE RENEWING FLOW

We consider the following velocity field $\mathbf{u} = (u_x, u_y)$ on a periodic square box $\Omega = [-L/2, L/2]^2$:

$$\begin{cases} u_x = U\sqrt{2(1-C)}\cos(ky + \phi_y) + U\sqrt{2C}\cos(kx + \phi_x), \\ u_y = 0, \end{cases} \quad 2nT \leq t < (2n+1)T \quad (1)$$

and

$$\begin{cases} u_x = 0, \\ u_y = U\sqrt{2(1-C)}\cos(kx + \phi_x) + U\sqrt{2C}\cos(ky + \phi_y), \end{cases} \quad (2n+1)T \leq t < 2(n+1)T, \quad (2)$$

where $n \in \mathbb{N}$, $k = 2\pi/L$, $U = \sqrt{\langle u^2 \rangle}$ is the root-mean-square velocity, and $C = \langle (\nabla \cdot \mathbf{u})^2 \rangle / \langle \|\nabla \mathbf{u}\|_F^2 \rangle$ is the degree of compressibility of the flow. Here, $\|\cdot\|_F$ is the Frobenius norm and $\langle \cdot \rangle$ denotes a spatial average over the domain Ω .

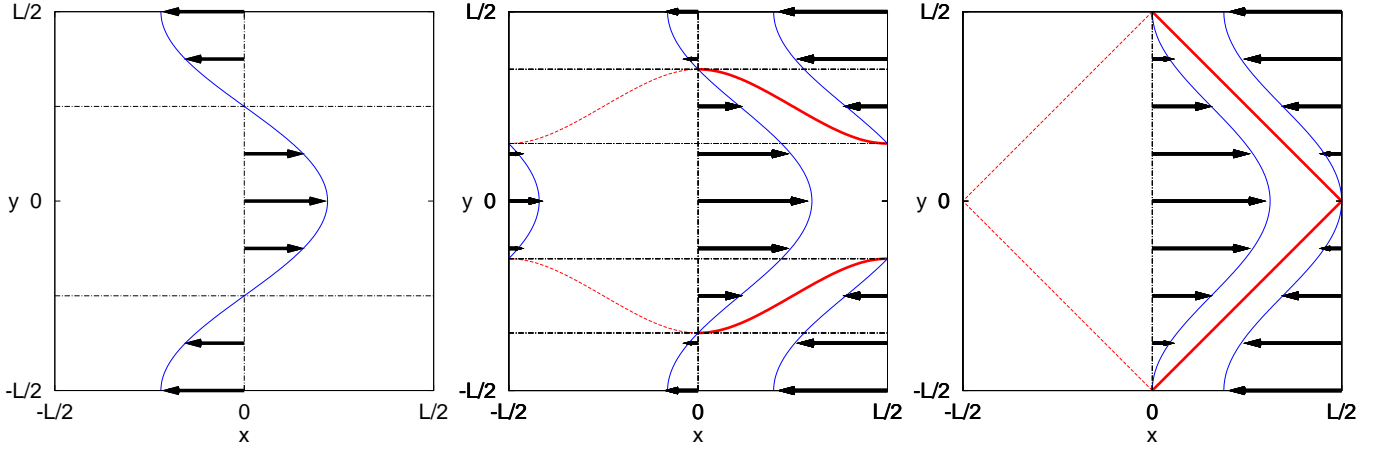


FIG. 2: (Colour online) Velocity profile of the renewing flow for $C = 0$ (left), $C = 1/4$ (middle) and $C = 1/2$ (right). The solid red lines are the portions of the stagnation lines where $\nabla \cdot \mathbf{u} < 0$.

Note that $0 \leq C \leq 1$; $C = 0$ corresponds to an incompressible flow, whereas $C = 1$ corresponds to a gradient flow. The angles ϕ_x and ϕ_y are independent random numbers uniformly distributed over $[0, 2\pi]$ and change randomly at each time period T .

The velocity field defined in (1) and (2) is a sequence of randomly translated sinusoidal profiles, each of which persists for a time T ; the velocity is alternatively oriented in the x and y directions. The renewing flow is in principle non-stationary, because the values of the velocity at two different times are either correlated or independent depending on whether or not the two times belong to the same frozen phase. However, it can be regarded as stationary for times much longer than T [42]. The temporal statistics of the flow is characterised by the correlation function:

$$F_{\mathcal{E}}(t) \equiv \frac{\langle \mathbf{u}(\mathbf{x}, s+t) \cdot \mathbf{u}(\mathbf{x}, s) \rangle_{\mathcal{E}}}{U^2} = \begin{cases} 1 - \frac{t}{T}, & t \leq T, \\ 0 & t > T, \end{cases} \quad (3)$$

where $\langle \cdot \rangle_{\mathcal{E}}$ denotes a space-time Eulerian average: $\langle \cdot \rangle_{\mathcal{E}} \equiv T^{-1} L^{-2} \int_0^T \int_{\Omega} \cdot \, ds d\mathbf{x}$. The correlation time of the flow is: $T_{\mathcal{E}} \equiv \int_0^T F_{\mathcal{E}}(t) dt = T/2$. A dimensionless measure of the correlation time is given by the Kubo number $Ku \equiv T U k$; Ku is proportional to the ratio of $T_{\mathcal{E}}$ and the eddy turnover time of the flow.

The position of a tracer evolves according to the following equation:

$$\dot{\mathbf{X}}(t) = \mathbf{u}(\mathbf{X}(t), t). \quad (4)$$

Figure 1 shows the distribution of tracers in the weakly compressible regime. The transition from the regime of weak compressibility to that of strong compressibility occurs when the maximum Lyapunov exponent of the flow becomes negative. In this case, the flow is not chaotic anymore and tracers are attracted to a pointlike set. To study this transition, it is useful to consider the set of stagnation points of the flow, in which $\mathbf{u} = (0, 0)$. Tracers indeed tend to accumulate in the neighbourhood of these points. From (1) and (2), we deduce that the set of stagnation points of the renewing flow consists of the two lines:

$$y = \pm \frac{1}{k} \arccos \left(\sqrt{\frac{C}{1-C}} \cos(kx + \phi_x) \right) - \frac{\phi_y}{k} + 2\pi m, \quad (5)$$

if the velocity is given by (1), or

$$y = \pm \frac{1}{k} \arccos \left(\sqrt{\frac{1-C}{C}} \cos(kx + \phi_x) \right) - \frac{\phi_y}{k} + 2\pi m, \quad (6)$$

if the velocity is given by (2). In (5) and (6), $m \in \mathbb{Z}$ is such that $(x, y) \in \Omega$. The stagnation lines are shown in figure 2 for some representative values of the degree of compressibility. If $C = 0$, the flow consists of parallel periodic ‘channels’ of width $L/2$. If $0 < C < 1/2$, the stagnation lines form barriers that block the motion of tracers over a portion of the domain whose size increases as C approaches $1/2$; the width of the periodic channels shrinks accordingly. Finally,

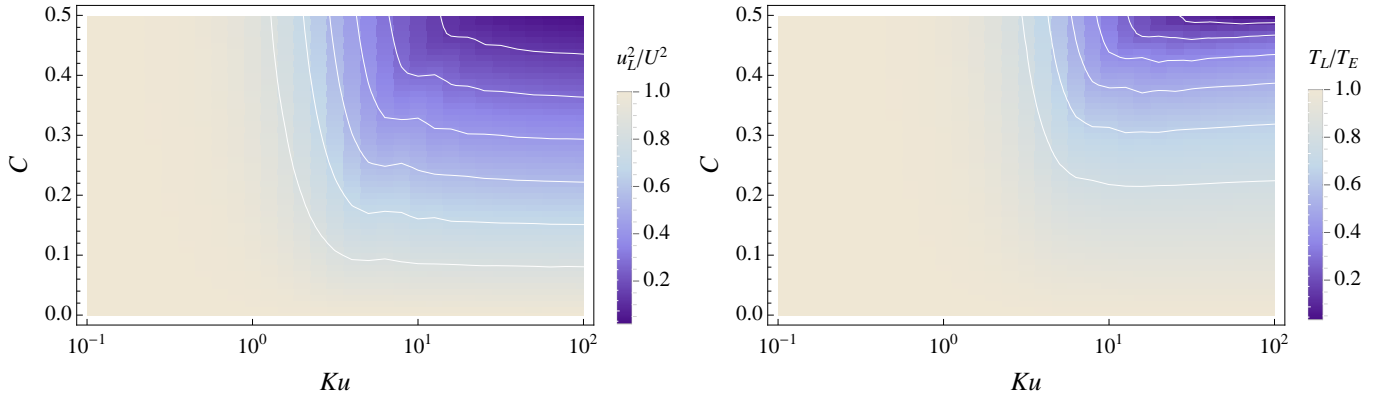


FIG. 3: (Colour online) Left: Ratio of the mean-square Lagrangian and Eulerian velocities as a function of the Kubo number Ku and of the degree of compressibility C of the flow. Right: Ratio of the Lagrangian and Eulerian correlation times of the velocity as a function of C and Ku .

if $1/2 \leq C \leq 1$, the stagnation lines divide the domain into regions that are not linked by any streamline. For these values of C , there are no periodic trajectories and if Ku is sufficiently large, all tracers collapse onto the stagnation lines. We conclude that the transition from the regime of weak clustering to that of strong clustering must occur for $C \leq 1/2$. However, the critical value of the degree of compressibility depends on Ku .

III. LAGRANGIAN VERSUS EULERIAN STATISTICS

The properties of the flow introduced in § II, namely the root-mean-square velocity, the degree of compressibility and the correlation function, are of an Eulerian nature. If the flow is compressible and has a nonzero correlation time, the Lagrangian counterparts of the aforementioned quantities may be different. Indeed, tracers are attracted towards the stagnation points and hence do not sample the phase space uniformly.

We define the root-mean-square Lagrangian velocity as: $u_{\mathcal{L}} \equiv \sqrt{\langle u^2(\mathbf{X}(s), s) \rangle_{\mathcal{L}}}$, where $u = |\mathbf{u}|$ and $\langle \cdot \rangle_{\mathcal{L}}$ is a Lagrangian average over both the random trajectory $\mathbf{X}(s)$ and time. Figure 3 (left panel) compares $u_{\mathcal{L}}^2$ and its Eulerian counterpart U^2 for different values of Ku and C . The plot includes 51 values of C between 0 and 1/2 and 31 values of Ku ranging from 10^{-1} to 10^2 . For each couple (Ku, C) , we have computed $u_{\mathcal{L}}^2$ by solving (4) for 10^2 tracers and for an integration time $t = 2 \times 10^3 T$ (to integrate (4), we have used a fourth-order Runge–Kutta method). At $C = 0$, $u_{\mathcal{L}}$ is the same as U for all values of Ku , because tracers explore the phase space uniformly. Likewise, at $Ku = 0$ the Eulerian and the Lagrangian statistics coincide independently of the value of C . By contrast, for $C \neq 0$ and $Ku \neq 0$, $u_{\mathcal{L}} < U$ because tracers are attracted towards the stagnation lines, where $u = 0$. The ability of the stagnation points to trap tracers strengthens with increasing C and Ku ; hence $u_{\mathcal{L}}$ eventually approaches zero.

The Lagrangian correlation function of the velocity is defined as:

$$F_{\mathcal{L}}(t) \equiv \frac{\langle \mathbf{u}(\mathbf{X}(s+t), s+t) \cdot \mathbf{u}(\mathbf{X}(s), s) \rangle_{\mathcal{L}}}{u_{\mathcal{L}}^2}. \quad (7)$$

The associate Lagrangian correlation time is: $T_{\mathcal{L}} \equiv \int_0^T F_{\mathcal{L}}(t) dt$. If the flow is incompressible ($C = 0$) or if it is decorrelated in time ($Ku = 0$), then $T_{\mathcal{L}} = T_{\mathcal{E}}$ (figure 3, right panel). If both C and Ku are nonzero, $T_{\mathcal{L}}$ is smaller than $T_{\mathcal{E}}$ and the ratio $T_{\mathcal{L}}/T_{\mathcal{E}}$ decreases as Ku or C increase (figure 3). Again, this behaviour can be explained by considering that a tracer is attracted towards the stagnation points, where $u = 0$, and hence its velocity decorrelates from the velocity it had at the beginning of the period. As Ku and C increase, a larger fraction of tracers get close to the stagnation points, and therefore the decorrelation is faster. The inset of figure 4 (left panel) also shows that, for C close to 1/2, $T_{\mathcal{L}}$ is proportional to T for small values of Ku , whereas it saturates to a value proportional to the eddy turnover time for large values of Ku . Indeed, after that time most of the tracers have reached a stagnation point, and their velocities have completely decorrelated.

Figure 4 (right panel) shows that, for small values of Ku , $F_{\mathcal{L}}(t)$ is the same as $F_{\mathcal{E}}(t)$ irrespective of the value of C . However, at large Ku , not only the Lagrangian correlation time of the velocity, but also the functional form of $F_{\mathcal{L}}(t)$ varies with C .

The degree of compressibility experienced by tracers also depends on Ku and C and differs from its Eulerian value if $Ku \neq 0$ and $C \neq 0$. Let us define the Lagrangian degree of compressibility as: $C_{\mathcal{L}} \equiv \langle (\nabla \cdot \mathbf{u})^2 \rangle_{\mathcal{L}} / \langle \|\nabla \mathbf{u}\|_F^2 \rangle_{\mathcal{L}}$. Then,

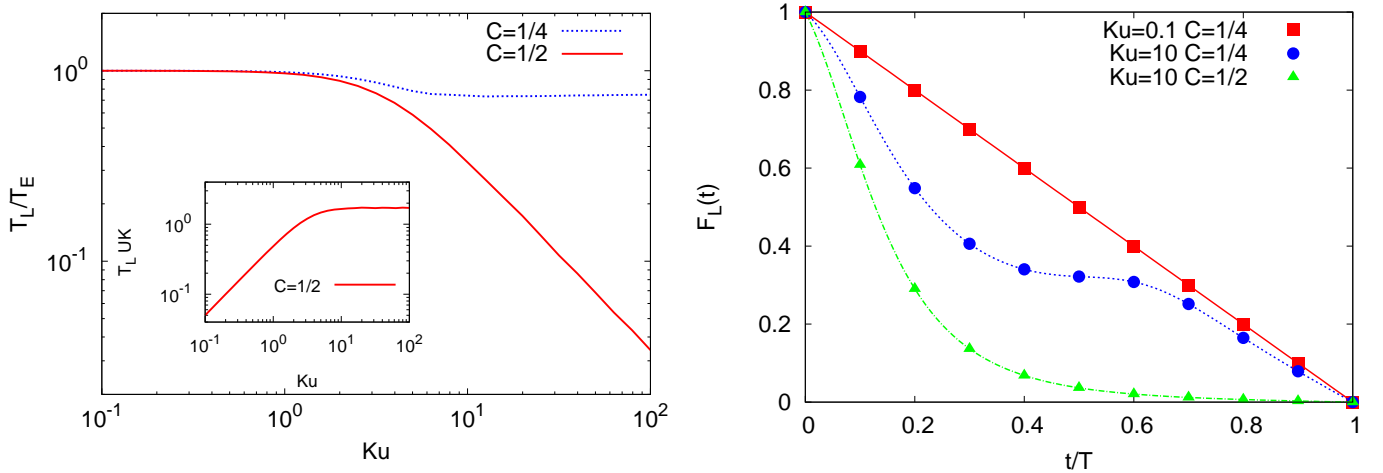


FIG. 4: (Colour online) Left: Ratio of the Lagrangian and Eulerian correlation times of the velocity as a function of Ku for $C = 1/4$ (dotted, blue curve) and $C = 1/2$ (solid, red curve); the inset shows the Lagrangian correlation time rescaled by $(Uk)^{-1}$ as a function of Ku for $C = 1/2$. Right: Lagrangian correlation function of the velocity for $Ku = 0.1$, $C = 1/4$ (squares), $Ku = 10$, $C = 1/4$ (circles) and $Ku = 10$, $C = 1/2$ (triangles).

$C_L > C$ for all nonzero values of Ku and C , because tracers spend more time in high-compressibility regions. For fixed Ku , the increase in compressibility is an increasing function of C , whereas for fixed C it is maximum when Ku is near to 1 and vanishes both in the small- and in the large- Ku limits (figure 5, left panel). The non-monotonic behaviour of the Lagrangian compressibility as a function of Ku is due to a peculiar feature of the model flow considered here. The stagnation lines (5) and (6) toward which the tracers are attracted do not coincide with the regions where the local compressibility of the flow is maximum, i.e., the lines $ky + \phi_y = m\pi$ for $2nT \leq t < (2n+1)T$ and $kx + \phi_x = m\pi$ for $(2n+1)T \leq t < 2(n+1)T$. Therefore, in the long-correlated limit the preferential sampling of the regions of strong compressibility is reduced.

IV. FRACTAL CLUSTERING

The spatial distribution of tracers within a fluid can be characterised in terms of the Lyapunov dimension [e.g. 36]:

$$D_L = N + \frac{\sum_{i=1}^N \lambda_i}{|\lambda_{N+1}|}, \quad (8)$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$ are the Lyapunov exponents of the flow and N is the maximum integer such that $\sum_{i=1}^N \lambda_i \geq 0$ (d denotes the dimension of the flow). Three different regimes can be identified. If the flow is incompressible ($\sum_{i=1}^d \lambda_i = 0$), tracers spread out evenly within the fluid ($D_L = d$). In the weakly compressible regime ($\sum_{i=1}^d \lambda_i < 0$ and $\lambda_1 > 0$), tracers cluster over a fractal set ($1 < D_L < d$). In the strongly compressible regime ($\sum_{i=1}^d \lambda_i < 0$ and $\lambda_1 < 0$), tracers are attracted to a pointlike set ($D_L = 0$). The transition from the regime of weak compressibility to that of strong compressibility occurs when λ_1 changes sign.

For the smooth d -dimensional Kraichnan [23] flow, the Lyapunov exponents can be calculated exactly (we remind the reader that in the Kraichnan model the velocity field is Gaussian, delta-correlated in time, and statistically homogenous and isotropic). The Lyapunov exponents are: $\lambda_i = \mathcal{D}\{d(d-2i+1) - 2C[d + (d-2)i]\}$, where $i = 1, \dots, d$, C is defined as in § II, and $\mathcal{D} > 0$ determines the amplitude of the fluctuations of the velocity gradient [26, 27]. Thus, for $d = 2$, the Lyapunov dimension of the smooth Kraichnan model is $D_L^0 = 2/(1+2C)$ and the weak-strong clustering transition occurs at $C = 1/2$. In time-correlated flows, the prediction of the Kraichnan model is recovered in the small- Ku limit [6, 22].

In this section, we study how the weak-strong clustering transition depends on Ku in the compressible renewing flow. To compute the Lyapunov exponents, we have used the method proposed by [5]; we have set the integration time to $t = 10^6$ for all values of Ku and C in order to ensure the convergence of the stretching rates to their asymptotic values.

The maximum Lyapunov exponent decreases with increasing C (figure 5, right panel). Its behaviour as a function of Ku is different in the weakly compressible and in the strongly compressible regimes. For small values of C , λ_1 is

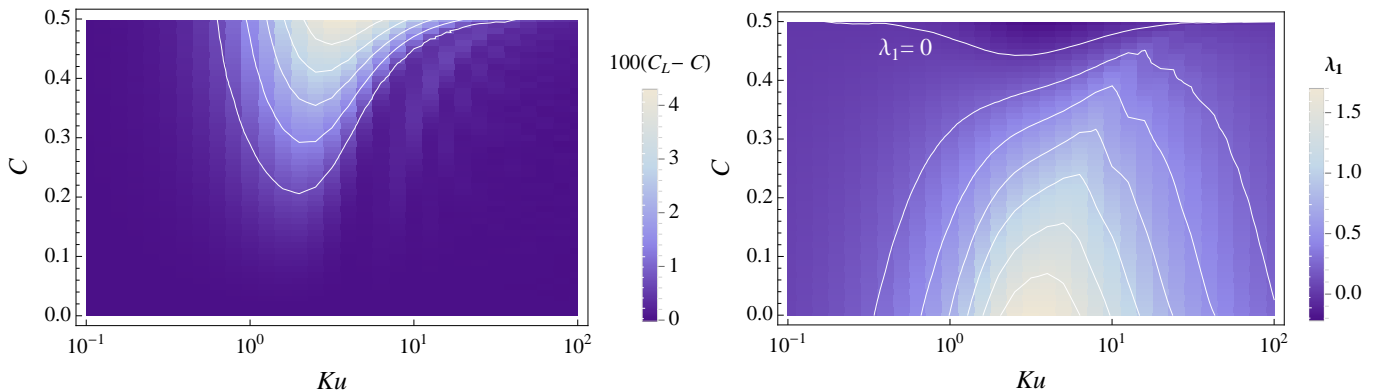


FIG. 5: (Colour online) Left: Increase in compressibility multiplied by 100, $(C_L - C) \times 100$, as a function of Ku and C . Right: Maximum Lyapunov exponent of the compressible renewing flow as a function of Ku and C

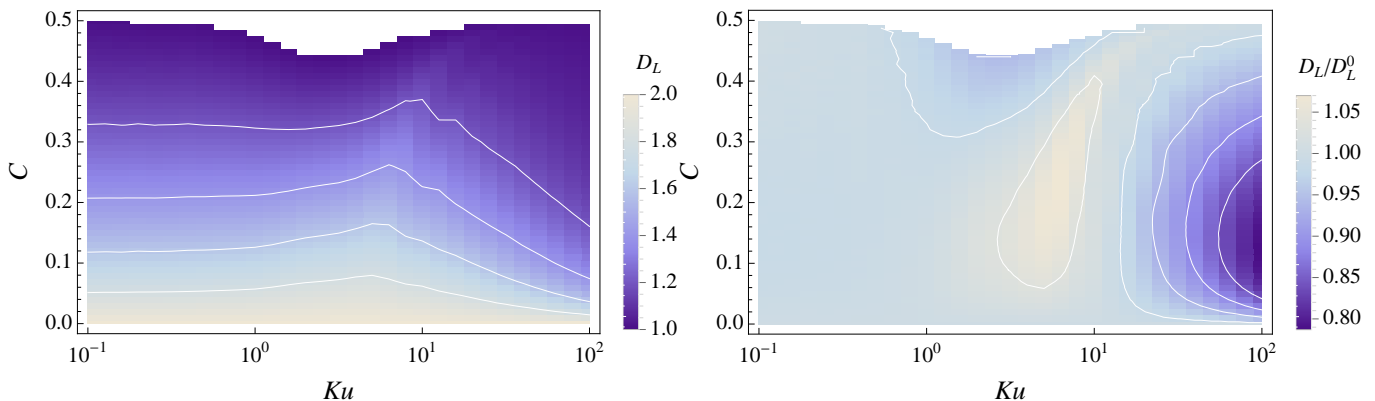


FIG. 6: (Colour online) Left: Lyapunov dimension D_L as a function of Ku and C . The region in which $D_L = 0$ is coloured in white in the plot. Right: D_L rescaled by its value at $Ku = 0$ as a function of Ku and C .

maximum for Ku near to 1, which signals an increase in chaoticity when the correlation time of the flow is comparable to the eddy turnover time. By contrast, for values of C near to $1/2$, λ_1 is minimum at $Ku \approx 1$, in accordance with the fact that the Lagrangian compressibility is maximum for these values of the parameters (figure 5). We also note that if Ku is near to 1, λ_1 becomes negative for $C < 1/2$; hence the weak-strong clustering transition occurs at a lower degree of compressibility compared to the short-correlated case.

Analogous conclusions can be reached by studying the behaviour of $D_L = 1 - \lambda_1/\lambda_2$ (figure 6). For fixed Ku , an increase in the Eulerian compressibility yields an increased level of clustering. The behaviour as a function of Ku is not monotonic and depends on the value of C . When Ku is near to 1, the level of clustering is minimum if C is small and is maximum if C is near to $1/2$; furthermore, D_L vanishes for values of C smaller than the critical value of the short-correlated case. The most important deviations of D_L from the $Ku = 0$ prediction are observed for values of Ku greater than 1 (figure 6, right panel).

V. CONCLUSIONS

We have studied the Lagrangian dynamics of tracers in a time-correlated compressible random flow as a function of the degree of compressibility and the Kubo number. The use of the compressible renewing flow has allowed us to examine a wide area of the parameter space (Ku, C) . We have shown that, in compressible random flows with nonzero correlation time, Lagrangian correlations differ significantly from their Eulerian counterparts, because tracers are attracted towards the stagnation points and therefore do not sample the phase space uniformly. This fact influences the spatial distribution of tracers within the fluid. In particular, in both the small- and the large- Ku limits, the critical degree of compressibility for the weak-strong clustering transition is the same as for a short-correlated flow. By contrast, when the correlation time of the flow is comparable to the eddy turnover time, a smaller degree of compressibility is required for the transition to occur. The non-monotonic behaviour of the critical

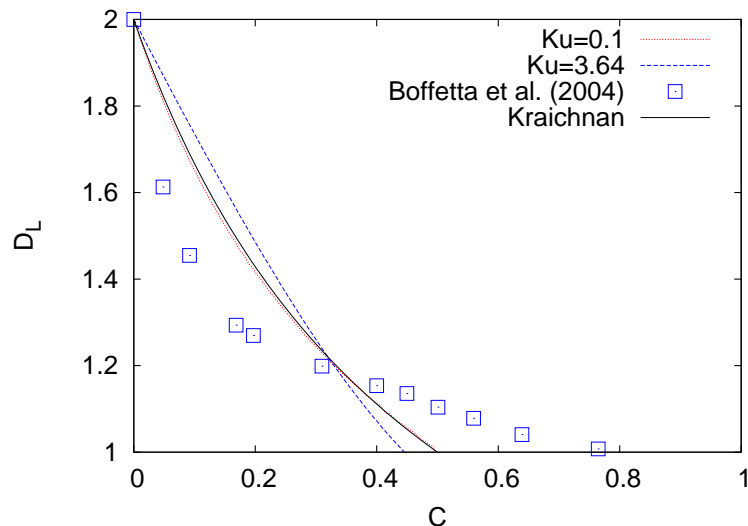


FIG. 7: (Colour online) D_L as a function of C for the Kraichnan model (solid, black curve) and for the compressible renewing flow with $Ku = 0.1$ (dotted, red curve) and $Ku = 3.64$ (dashed, blue curve). The square symbols are from the direct numerical simulation of a turbulent surface flow by Boffetta *et al.* [6].

degree of compressibility is a consequence of the fact that the stagnation points do not coincide with the points in which the compressibility is maximum. This behaviour is very different from that observed by Gustavsson & Mehlig [22] in a Gaussian velocity field with exponential spatio-temporal correlations. In that flow, the critical degree of compressibility indeed decreases monotonically as a function of Ku and tends to zero in the large- Ku limit, i.e. the weak-strong clustering transition is more and more favoured as Ku increases.

The comparison of our results for intermediate values of Ku with those by Boffetta *et al.* [6] reveals yet another difference. In the compressible renewing flow, the clustering is reduced compared to the small- Ku case if the compressibility is small, but it is enhanced if the compressibility is large. This behaviour is the opposite of that found in the turbulent surface flow (figure 7). Moreover, in the renewing flow, the critical degree of compressibility is less than or equal to $1/2$ for all values of Ku ; in the surface flow, it is significantly greater (figure 7). In the light of our findings, it would be interesting to examine the distribution of the stagnation points of the surface flow considered by [6] and understand how it influences the statistics of clustering.

In conclusion, the differences between our findings and those obtained in different flows demonstrate that the properties of tracer dynamics in time-correlated compressible flows are strongly non-universal, to the extent that flows with comparable C and Ku can have an opposite effect on clustering. In particular, the level of clustering depends dramatically on the peculiar structures of the velocity field toward which tracers are attracted.

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